ALMOST AUTOMORPHIC MILD SOLUTIONS TO FRACTIONAL PARTIAL DIFFERENCE-DIFFERENTIAL EQUATIONS

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Abstract. We study existence and uniqueness of almost automorphic solutions for nonlinear partial difference-differential equations modeled in abstract form as

$$(*) \qquad \Delta^\alpha u(n) = Au(n+1) + f(n, u(n)), \qquad n \in \mathbb{Z},$$

for $0 < \alpha \leq 1$ where A is the generator of a C_0-semigroup defined on a Banach space X, Δ^α denote fractional difference in Weyl-like sense and f satisfies Lipchitz conditions of global and local type. We introduce the notion of α-resolvent sequence $\{S_\alpha(n)\}_{n\in\mathbb{N}_0} \subset \mathcal{B}(X)$ and we prove that a mild solution of $(*)$ corresponds to a fixed point of

$$u(n+1) = \sum_{j=-\infty}^{n} S_\alpha(n-j) f(j, u(j)), \qquad n \in \mathbb{Z}.$$

We show that such mild solution is strong in case of the forcing term belongs to an appropriate weighted Lebesgue space of sequences. Application to a model of population of cells is given.

1. Introduction

Our concern in this paper is the study of the existence and uniqueness of almost automorphic solutions for the nonlinear fractional difference equation

$$\Delta^\alpha u(n) = Au(n+1) + f(n, u(n)), \qquad n \in \mathbb{Z}, \tag{1.1}$$

where A is a closed linear operator with domain $D(A)$ defined on a Banach space X and $0 < \alpha \leq 1$.

So far the study on fractional difference equations focuses in the discussion of different but related definitions that may allow nice properties of calculus [5, 20, 21, 26, 27, 1, 40], existence of solutions of different classes of nonlinear fractional difference equations and in the development of qualitative properties such as existence of positive solutions and geometrical properties [5, 23, 24]. In the last time, applications to concrete models have been analyzed [6, 7]. Nonetheless, all these studies are concentrated either in finite dimensional cases or, concerning (1.1), at most a bounded operator A, which resembles the study of ordinary differential equations. However, the fractional modeling of partial differential equations that have a mixed character, i.e. that can be modeled both in discrete time as well as in continuous space (or vice-versa), is an untreated topic that deserves to be investigated. We observe that a wide range of this class of models can be represented abstractly by (1.1), where A is an unbounded operator.

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For instance, abstract difference equations of the form (1.1) with $\alpha = 1$ appears naturally in traffic dynamics [13], [28]. In such case, the equation of motion of a line of identical vehicles is given by (1.1) where $A = \rho\frac{\partial}{\partial t}$ is a differential and, consequently, unbounded operator. Here $\rho > 0$ is a positive number that incorporates the information on the mass of each vehicle and the sensitivity of the control mechanism [13, p. 170]. In recent years, the significance of such models has increased and a detailed understanding of this key process is now becoming more important. We observe that, in appropriate spaces of functions, the differential operator $\rho\frac{\partial}{\partial t}$ is the generator of the well known C_0-semigroup of translations [19, Chapter II, Section 2.10].

It is noteworthy that mixed partial difference-differential equations occur not only in traffic dynamics, but also in the theory of probability and in the theory of the chain processes of chemistry and radioactivity [8, p.498]. Thus, the analysis of (1.1) for $0 < \alpha < 1$ and A being the generator of a C_0-semigroup should provide new insights of the discrete sub diffusive behavior of the equation because it incorporate memory effects of the materials used for each specific model.

On the other hand, we recall that the concept of almost automorphic functions was introduced by S. Bochner four decades ago. The interest in almost automorphic solutions of evolution equations and their applications has been an important topic of research in the last time. See [39], [31], [17] and references therein. New methods and new concepts have been introduced in the literature in the last decade. The range of applications include at present linear and nonlinear evolution equations, integro-differential and functional-differential equations, dynamical systems, and so on [30, 10, 33, 22, 44].

The study of almost automorphic solutions for difference equations constitutes a recent research area. This generalizes the analysis of almost periodic solutions for difference equations which have been more deeply investigated. Almost periodic sequences appeared early in the theory of almost periodic functions [14]. General results concerning almost periodic solutions for difference equations are available nowadays. For instance, we mention those related to positive solutions and asymptotic solutions [18, 36]. For more information, we refer to the monograph [14] and references therein. In contrast, the concept of almost automorphic sequence was introduced only recently by Minh, Naito and N'Guérékata in [38, Definition 2.6] and it was later contained in the works of Caraballo and Cheban [11], [12]. A first systematic study of their main properties listed in [4] and further generalizations and applications has been performed in [29], [16], [32] and [2], among others. Beyond that a discrete theory is strongly motivated by applications e.g. in population biology, in addition, it serves as a basic tool to understand numerical discretization and is often essential for the analysis of continuous problems. Concerning (1.1), almost automorphic functions could play the role of harmonic oscillations covered with big noise [45].

The first open problem in order to begin the study of the existence of almost automorphic solutions to (1.1) is to find an adequate notion of fractional difference operator. Indeed, there are multiple definitions of the fractional difference (both delta and nabla), which are well known to be translationally equivalent to one another [9]. Nonetheless, these definitions do have various peculiarities, which affect their use in the study of particular problems.

In this paper, we solve this problem, by proposing a definition of Weyl-like fractional difference by means of which the treatment of abstract fractional difference equations defined on the time scale $\mathbb{Z}$ seems to be feasible. In this way, our definition set at least one avenue of research, whose study is perhaps easier by the utilization of a modified definition of the fractional delta difference. Indeed, one of our main results show that our definition of Weyl-like fractional difference is consistent with the solution of (1.1) interpreted as a fixed point of the problem

$$u(n) = \sum_{j=-\infty}^{n-1} S_\alpha(n-1-j)f(j,u(j)), \qquad n \in \mathbb{Z} \tag{1.2}$$

where $S_\alpha(n)$ is a sequence of bounded and linear operators that commutes with A on $D(A)$ and satisfy the property

$$S_\alpha(n)x = k^\alpha(n)x + A(k^\alpha * S_\alpha)(n)x, \quad x \in X, \tag{1.3}$$

for all $n \in \mathbb{N}_0$, where

$$k^\alpha(n) := \frac{\Gamma(n+\alpha)}{\Gamma(\alpha)\Gamma(n+1)}. \tag{1.4}$$

Our second main contribution in this paper is the introduction of a method of analysis of existence and qualitative behavior of solutions for (1.1) by means of the sequence of operators (1.3). In fact, they play the role of the semigroup $T(t)$ generated by A in the solution formula $u(t) = \displaystyle\int_{-\infty}^{t} T(t-s)f(s,u(s))ds$ of the classical problem $u'(t) = Au(t) + f(t,u(t))$ for $t \in \mathbb{R}$. The sequence of operators $S_\alpha(n)$ has the advantage that it have the following representation:

$$S_\alpha(n)x = \sum_{j=1}^{n} \beta_{\alpha,n}(j)(I-A)^{-(j+1)}x, \quad n \in \mathbb{N}, \quad x \in X.$$

This is due to the fact that (1.3) implies that the operator $I - A$ must be invertible and that $S_\alpha(0) = (I-A)^{-1}$ in contrast with the case of the scale $\mathbb{R}$ where the behavior near to zero of the family of operators determined by the linear part of fractional differential equations is not well described, in general. This is explained, of course, by the discrete character of the scale $\mathbb{Z}$.

This paper is organized as follows: Section 2 is mainly devoted to recall the definition of almost automorphic sequence and to introduce the concept of Weyl-like fractional difference (Definition 2.3) by means of the notion of Weyl-like fractional sum. It incorporates the kernel (1.4) that seems to be at the basis of the discretization of the continuous fractional differential operator in the sense of Riemann-Liouville [15, 40, 34]. In Section 3 we introduce an operator theoretical method for the treatment of the linear part of (1.1) (Definition 3.1). It is interesting to note the regularized character of the introduced concept of operator resolvent sequence (Theorem 3.2). More importantly, whenever the operator A in (1.1) is the generator of a C_0-semigroup, an explicit form of such operator resolvent sequence can be given (Theorem 3.5). This characteristic allows an analysis of qualitative properties, like summability, which turns out to be important for the study of almost automorphic solutions of (1.1). Section 4 is concerned with the study of the non homogeneous linear difference equation

$$\Delta^\alpha u(n) = Au(n+1) + f(n), \quad n \in \mathbb{Z}.$$

The main objective is to prove the consistence of the given concept of Weyl-like fractional difference and the representation of the solution of (1.1) by means of the formula (1.2), see Theorem 4.2. Then, the notion of mild solution is introduced and our first result on existence of almost automorphic solution is given (Theorem 4.5). Section 5 is devoted to the study of almost automorphic solutions for the nonlinear equation (1.1). We present two theorems of existence and uniqueness of almost automorphic solutions based on the Banach fixed point theorem, that consider either global as well as local Lipschitz type conditions on the nonlinearity $f(n,u)$ (see Theorems 5.2 and 5.4). Finally, the last section 6 provides some examples to illustrate how the abstract results of the previous sections apply. Our first example brings into consideration a model that describes the number of a population of cells distinguished by their individual size. We present their abstract formulation and we give new insights on existence and uniqueness of almost automorphic solutions for the fractional model.

2. Preliminaries

Let X be a complex Banach space. We denote by $s(\mathbb{Z},X)$ the vector space consisting of all vector-valued sequences $f:\mathbb{Z}\to X$. Let $\rho:\mathbb{Z}\to(0,\infty)$ be a a positive sequence (weight). Let $1\le p<\infty$ be given. By $l^p_\rho(\mathbb{Z},X)$ we denote the set of vector-valued sequences $f:\mathbb{Z}\to X$ such that

$$\|f\|_{l^p_\rho}:=\sum_{n=-\infty}^{\infty}\|f(n)\|^p\rho(n)<\infty.$$

When $X=\mathbb{C}$ we write $s(\mathbb{Z})$ and $l^p_\rho(\mathbb{Z})$ respectively. Moreover, if $\rho(n)\equiv 1$ then we write $l^p(\mathbb{Z})$. We define the forward Euler operator $\Delta: s(\mathbb{Z},X)\to s(\mathbb{Z},X)$ given by

$$\Delta f(n)=f(n+1)-f(n),\qquad n\in\mathbb{Z}.$$

Recursively we define

$$\Delta^{k+1}=\Delta^k\Delta=\Delta\Delta^k,\qquad k\in\mathbb{N},$$

and $\Delta^0=I$ is the identity operator. It is easy to see that

$$\Delta^k f(n)=\sum_{j=0}^{k}(-1)^{k-j}\binom{k}{j}f(n+j).$$

In particular $\Delta^1=\Delta$. In addition, for $\alpha>0$, we consider the scalar sequence $\{k^\alpha(n)\}_{n\in\mathbb{N}_0}$ defined by

$$k^\alpha(n):=\frac{\Gamma(n+\alpha)}{\Gamma(\alpha)\Gamma(n+1)}.$$

We note that the kernel k^α satisfies the semigroup property in $\mathbb{N}_0$, that is,

$$(k^\alpha * k^\beta)(n)=\sum_{j=0}^{n}k^\alpha(n-j)k^\beta(j)=k^{\alpha+\beta}(n)$$

with $n\in\mathbb{N}_0$ and $\alpha,\beta>0$. This property has been observed in [9] and follows easily by noting that the kernel $k^\alpha(n)$ can be equivalently defined by means of the generating function

$$\sum_{n=0}^{\infty}k^\alpha(n)z^n=\frac{1}{(1-z)^\alpha},\quad |z|<1,$$

see [47, Vol I, p.77]. Furthermore, the following equality holds: for $\alpha>0$,

$$k^\alpha(n)=\frac{n^{\alpha-1}}{\Gamma(\alpha)}(1+O(\frac{1}{n})),\qquad n\in\mathbb{N},$$

([47, Vol. I, p.77 (1.18)]) and k^α is increasing (as a function of n) for $\alpha>1$, decreasing for $0<\alpha<1$ and $k^1(n)=1$ for $n\in\mathbb{N}$ ([47, Theorem III.1.17]). We note that $k^\alpha(n)$ is in agreement with the kernel used in the definition of discrete fractional derivative for $\alpha<0$ and step $h=1$ given recently in [40, formula (27)]. The above considerations suggest the following definition.

Definition 2.1. *Let $\alpha > 0$ be given and $\rho(n) = |n|^{\alpha-1}, n \in \mathbb{Z}$. The α-th fractional sum of a sequence $f \in l^1_\rho(\mathbb{Z}, X)$ is defined by*

$$\Delta^{-\alpha} f(n) := \sum_{j=-\infty}^{n} k^\alpha(n-j) f(j), \qquad n \in \mathbb{Z}.$$

See also [35] for related work on a slight variant of this definition.

Remark 2.2. *The previous definition can be numerically compared with the continuous fractional integral in the sense of Weyl, see* [40, Section 3.3]. *Moreover, we observe that as a consequence of the semigroup property of the kernel k^α we have that $\Delta^{-\alpha}\Delta^{-\beta} = \Delta^{-(\alpha+\beta)} = \Delta^{-\beta}\Delta^{-\alpha}$.*

In what follows, we always denote $\rho(n) = |n|^{\alpha-1}, \quad n \in \mathbb{Z}$.

Definition 2.3. *Let $\alpha > 0$ be given. The α-th fractional difference of a sequence $f \in l^1_\rho(\mathbb{Z}, X)$ is defined by*

$$\Delta^\alpha f(n) := \Delta^m \Delta^{-(m-\alpha)} f(n), \qquad n \in \mathbb{Z},$$

with $m = [\alpha] + 1$.

Remark 2.4. *Note that if $f \in l^1_\rho(\mathbb{Z}, X)$ then*

$$\begin{aligned}
\Delta^{-(m-\alpha)}\Delta^m f(n) &= \sum_{j=-\infty}^{n} k^{m-\alpha}(n-j)\Delta^m f(j) \\
&= \sum_{j=-\infty}^{n} k^{m-\alpha}(n-j)\sum_{i=0}^{m}(-1)^{m-i}\binom{m}{i} f(j+i) \\
&= \sum_{i=0}^{m}(-1)^{m-i}\binom{m}{i}\sum_{j=-\infty}^{n} k^{m-\alpha}(n-j) f(j+i) \\
&= \sum_{i=0}^{m}(-1)^{m-i}\binom{m}{i}\sum_{u=-\infty}^{n+i} k^{m-\alpha}(n+i-u) f(u) \\
&= \sum_{i=0}^{m}(-1)^{m-i}\binom{m}{i}\Delta^{-(m-\alpha)} f(n+i) = \Delta^m\Delta^{-(m-\alpha)} f(n),
\end{aligned}$$

where we have applied the Fubini's Theorem and a change of variable. In other words, the discrete notions of fractional difference when defined either in the sense of Caputo or Riemann-Liouville, coincide.

For a sequence defined on $\mathbb{N}_0$, the theory and applications of fractional differences defined by means of the kernel $k^\alpha(n)$ is developed in [9]. Furthermore, it is clear from Definition 2.3 that $\lim_{\alpha\to 0} \Delta^\alpha f(n) = f(n) = \lim_{\alpha \to 0} \Delta^{-\alpha} f(n)$ for $f \in l^1_\rho(\mathbb{Z}, X)$, with $\alpha > 0$.

Now we recall the notion of almost automorphic sequences. A sequence $f : \mathbb{Z} \to X$ is called almost automorphic if for every integer sequence $\{k'_n\}$, there exists a subsequence $\{k_n\}$ such that

$$\bar{f}(k) := \lim_{n\to\infty} f(k + k_n)$$

is well defined for each $k \in \mathbb{Z}$ and $\lim_{n\to\infty} \bar{f}(k - k_n) = f(k)$, see [4, Definition 2.1] and references therein. We denote by $AA_d(\mathbb{Z}, X)$ the set of almost automorphic sequences. It is well known that the set $AA_d(\mathbb{Z},X)$ endowed with the norm $\|f\|_\infty := \sup_{k\in\mathbb{Z}}\|f(k)\|$ is a Banach space, see [4, Theorem 2.4]. A typical example is

$$f(k) = \sin\left(\frac{1}{2+\cos(k)+\cos(\sqrt{2}k)}\right), \quad k \in \mathbb{Z}.$$

Throughout this paper, we use several properties about almost automorphic sequences that appear in [4]. In particular, we will need the following theorem.

Theorem 2.5. [4, Theorem 2.10] *Let $u : \mathbb{Z} \to X$ be a discrete almost automorphic sequence and let $f : \mathbb{Z}\times X \to X$ be a discrete almost automorphic function in $k \in \mathbb{Z}$ for each $x \in X$ that satisfies a global Lipschitz condition in $x \in X$ uniformly in $k \in \mathbb{Z}$, that is, there is a constant $L > 0$ such that $\|f(k,x)-f(k,y)\| \le L\|x-y\|$, for all $x, y \in X, k \in \mathbb{Z}$ then, the Nemytskii operator $U : \mathbb{Z} \to X$ defined by $U(k) = f(k,u(k))$ is discrete almost automorphic.*

The previous theorem admits a new version with local conditions on the function f, see [3].

Corollary 2.6. *Let $f : \mathbb{Z}\times X \to X$ be a discrete almost automorphic function in $k \in \mathbb{Z}$ for each $x \in X$ that satisfies a local Lipschitz condition that is, for each positive number r, for all $k \in \mathbb{Z}$ and for all $x, y \in X$ with $\|x\| \le r$ and $\|y\| \le r$, we have $\|f(k,x)-f(k,y)\| \le L(r)\|x-y\|$, where $L : \mathbb{R}_+ \to \mathbb{R}_+$ is a nondecreasing function. Then, the conclusion of the previous theorem is true.*

We will need the following function, called stable Lévy process,

$$f_{t,\alpha}(\lambda) = \frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} e^{z\lambda - tz^\alpha}dz, \quad \sigma > 0, \quad t > 0, \quad \lambda \ge 0, \quad 0 < \alpha < 1, \tag{2.1}$$

where the branch of z^α is so taken that $Re(z^\alpha) > 0$ for $Re(z) > 0$. This branch is single-valued in the z-plane cut along the negative real axis.

Proposition 2.7. *The following properties hold:*

(i) $\displaystyle\int_0^\infty e^{-\lambda a} f_{t,\alpha}(\lambda)d\lambda = e^{-ta^\alpha}, \quad t > 0, \quad a > 0.$

(ii) $f_{t,\alpha}(\lambda) \ge 0, \quad \lambda > 0.$

(iii) $\displaystyle\int_0^\infty f_{t,\alpha}(\lambda)d\lambda = 1.$

For a proof, see [46, p.260-262]. We also recall that the Mittag-Leffler function is defined as follows

$$E_{\alpha,\beta}(z) := \sum_{n=0}^\infty \frac{z^n}{\Gamma(\alpha n+\beta)}, \quad \alpha > 0, \quad \beta > 0, \quad z \in \mathbb{C}.$$

One of the most interesting properties is associated with their Laplace transform:

$$\int_0^\infty e^{-\lambda t} t^{\beta-1} E_{\alpha,\beta}(\pm\omega t^\alpha)dt = \frac{\lambda^{\alpha-\beta}}{\lambda^\alpha \mp \omega}, \quad Re(\lambda) > |\omega|^{1/\alpha}, \tag{2.2}$$

see [42, Section 1.2 , formula (1.80)] and with their asymptotic behavior: If $0 < \alpha < 2$, $\beta > 0$, then

$$E_{\alpha,\beta}(z) = \frac{1}{\alpha} z^{\frac{1-\beta}{\alpha}} exp(z^{1/\alpha}) + \epsilon_{\alpha,\beta}(z), \quad |arg(z)| \le \frac{1}{2}\alpha\pi$$

where

$$\epsilon_{\alpha,\beta}(z) = -\sum_{n=1}^{N-1} \frac{z^{-n}}{\Gamma(\beta - \alpha n)} + O(|z|^N) \quad (z \to \infty).$$

For more details on the Mittag-Leffler function see [37, Appendix E].

We also recall the function

$$g_\alpha(t) = \frac{t^{\alpha-1}}{\Gamma(\alpha)}, \quad \alpha > 0.$$

For further use, we also introduce the following definition.

Definition 2.8. *An operator-valued sequence $\{S(n)\}_{n\in\mathbb{N}_0} \subset \mathcal{B}(X)$ is called summable if*

$$\|S\|_1 := \sum_{n=0}^{\infty} \|S(n)\| < \infty.$$

3. Operator resolvent sequences

In this section, we introduce the notion of α-resolvent family of bounded and linear operators. This concept will be useful in the treatment of fractional difference equations as we will see in the next section. Moreover, the knowledge of the properties of the family of bounded operators provide insights on the qualitative behavior of the solutions of fractional difference equations.

Definition 3.1. *Let $\alpha > 0$ and A be a closed linear operator with domain $D(A)$ defined on a Banach space X. An operator-valued sequence $\{S_\alpha(n)\}_{n\in\mathbb{N}_0} \subset \mathcal{B}(X)$ is called a discrete α-resolvent family generated by A if it satisfies the following conditions*

(i) *$S_\alpha(n)Ax = AS_\alpha(n)x$ for $n \in \mathbb{N}_0$ and $x \in D(A)$;*

(ii) *$S_\alpha(n)x = k^\alpha(n)x + A(k^\alpha * S_\alpha)(n)x$, for all $n \in \mathbb{N}_0$ and $x \in X$.*

For $\alpha > 0$ fixed and each $n \in \mathbb{N}$ we define the sequence $\{\beta_{\alpha,n}(j)\}_{j=1,\dots,n}$ as follows:

For $n = 1$,

$$\beta_{\alpha,1}(1) = k^\alpha(1).$$

For $n = 2$,

$$\beta_{\alpha,2}(1) = k^\alpha(2) - k^\alpha(1)\beta_{\alpha,1}(1),$$

and

$$\beta_{\alpha,2}(2) = k^\alpha(1)\beta_{\alpha,1}(1).$$

For $n = 3$,

$$\beta_{\alpha,3}(1) = k^\alpha(3) - k^\alpha(2)\beta_{\alpha,1}(1) - k^\alpha(1)\beta_{\alpha,2}(1),$$
$$\beta_{\alpha,3}(2) = k^\alpha(2)\beta_{\alpha,1}(1) + k^\alpha(1)\beta_{\alpha,2}(1) - k^\alpha(1)\beta_{\alpha,2}(2)$$

and

$$\beta_{\alpha,3}(3) = k^\alpha(1)\beta_{\alpha,2}(2).$$

For $n \geq 4$,

$$\beta_{\alpha,n}(1) = k^\alpha(n) - \sum_{j=1}^{n-1} k^\alpha(n-j)\beta_{\alpha,j}(1),$$

$$\beta_{\alpha,n}(l) = \sum_{j=l-1}^{n-1} k^\alpha(n-j)\beta_{\alpha,j}(l-1) - \sum_{j=l}^{n-1} k^\alpha(n-j)\beta_{\alpha,j}(l) \quad 2 \leq l \leq n-1,$$

and

$$\beta_{\alpha,n}(n) = k^\alpha(1)\beta_{\alpha,n-1}(n-1).$$

The following result shows some properties of α-resolvent families concerning their regularity, and that they have always an explicit representation in terms of a bounded linear operator.

Theorem 3.2. *Let $\{S_\alpha(n)\}_{n\in\mathbb{N}_0} \subset \mathcal{B}(X)$ be a discrete α-resolvent family generated by A. Then the following statements holds:*

(i) $1 \in \rho(A)$.

(ii) *for all $x \in X$ we have that $S_\alpha(0)x = (I-A)^{-1}x$ and*

$$S_\alpha(n)x = \sum_{j=1}^{n} \beta_{\alpha,n}(j)(I-A)^{-(j+1)}x, \quad n \in \mathbb{N}.$$

(iii) *for all $x \in X$ we have that $S_\alpha(0)x \in D(A)$ and $S_\alpha(n)x \in D(A^2)$ for all $n \in \mathbb{N}$.*

Proof. Note that by Definition 3.1 part (ii) we have that

$$S_\alpha(0)x = x + AS_\alpha(0), \qquad x \in X.$$

Then

$$(I-A)S_\alpha(0)x = x \qquad x \in X,$$

and using Definition 3.1 part (i) we get that

$$S_\alpha(0)(I-A)x = x, \qquad x \in D(A),$$

and we conclude that $1 \in \rho(A)$ and $S_\alpha(0)x = (I-A)^{-1}x$ for all $x \in X$.

Now let $x \in X$. We have that

$$S_\alpha(1)x = k^\alpha(1)x + k^\alpha(1)AS_\alpha(0)x + AS_\alpha(1)x,$$

then

$$(I-A)S_\alpha(1)x = k^\alpha(1)\Big(I + A(I-A)^{-1}\Big)x = k^\alpha(1)(I-A)^{-1}x,$$

where we have used that $S_\alpha(0) = (I-A)^{-1}$ and $A(I-A)^{-1} = (I-A)^{-1} - I$, and therefore

$$S_\alpha(1)(I-A)x = k^\alpha(1)(I-A)^{-1}x, \quad x \in D(A).$$

Then $S_\alpha(1)x = k^\alpha(1)(I-A)^{-2}x = \beta_{\alpha,1}(1)(I-A)^{-2}x$. By induction, we suppose that

$$S_\alpha(j)x = \sum_{i=1}^{j} \beta_{\alpha,j}(i)(I-A)^{-(i+1)}x$$

for $j \le n-1$. We write

$$\begin{aligned} S_\alpha(n)x &= k^\alpha(n)x + A\sum_{j=0}^{n} k^\alpha(n-j)S_\alpha(j)x \\ &= k^\alpha(n)x + A\sum_{j=1}^{n-1} k^\alpha(n-j)S_\alpha(j)x + k^\alpha(n)AS_\alpha(0)x + AS_\alpha(n)x. \end{aligned}$$

Then

$$\begin{aligned}
(I-A)S_\alpha(n)x &= k^\alpha(n)(I+AS_\alpha(0))x + A\sum_{j=1}^{n-1}k^\alpha(n-j)S_\alpha(j)x\\
&= k^\alpha(n)(I+A(I-A)^{-1})x + A\sum_{j=1}^{n-1}k^\alpha(n-j)\sum_{i=1}^{j}\beta_{\alpha,j}(i)(I-A)^{-(i+1)}x\\
&= k^\alpha(n)(I+A(I-A)^{-1})x + A\sum_{i=1}^{n-1}\eta_{\alpha,n}(i)(I-A)^{-(i+1)}x,
\end{aligned}$$

with $\eta_{\alpha,n}(i) := \sum_{j=i}^{n-1} k^\alpha(n-j)\beta_{\alpha,j}(i)$ for $i=1,\dots,n-1$. Using that $A(I-A)^{-1} = (I-A)^{-1} - I$, we get that

$$\begin{aligned}
(I-A)S_\alpha(n)x &= k^\alpha(n)(I-A)^{-1}x + \sum_{i=1}^{n-1}\eta_{\alpha,n}(i)\Big((I-A)^{-(i+1)} - (I-A)^{-i}\Big)x\\
&= \sum_{j=1}^{n}\beta_{\alpha,n}(j)(I-A)^{-j}x,
\end{aligned}$$

where $\beta_{\alpha,n}(n) = \eta_{\alpha,n}(n-1)$, $\beta_{\alpha,n}(j) = \eta_{\alpha,n}(j-1) - \eta_{\alpha,n}(j)$ for $j = 2,\dots,n-1$, and $\beta_{\alpha,n}(1) = k^\alpha(n) - \eta_{\alpha,n}(1)$. So for $x \in D(A)$ the identity

$$S_\alpha(n)(I-A)x = \sum_{j=1}^{n}\beta_{\alpha,n}(j)(I-A)^{-j}x$$

holds. Finally, observe that by part (ii) of the above result, for all $x \in X$ it is a straightforward consequence that $S_\alpha(0)x \in D(A)$ and $S_\alpha(n)x \in D(A^2)$ for $n \in \mathbb{N}$. Then we conclude the result.

□

Remark 3.3. *Observe that the property (ii) remember the solution of the equation* $\Delta u(n) = Au(n+1)$, $n \in \mathbb{N}_0$ *which is given by* $u(n) = (I-A)^{-n}u(0)$, $n \in \mathbb{N}_0$. *In other words, even when* A *is an unbounded operator we have that the discrete semigroup, namely* $(I-A)^{-n}$, *constitutes a sequence of bounded linear operators. This is an interesting feature that Definition 3.1 brings to the fractional case and therefore permits to treat the fractional homogeneous problem* $\Delta^\alpha u(n) = Au(n+1)$, $n \in \mathbb{N}_0$, $\alpha > 0$ *where* A *is unbounded, by means of a family of bounded and linear operators that can play the role of discrete semigroups.*

Remark 3.4. *We note that property (ii) indicates a better regularity that is not present in the continuous case. This is implied by the regularizing character of the property* $S_\alpha(0) = (I-A)^{-1}$, *which in turn responds to the discrete character of the scale* $\mathbb{Z}$.

Our next result gives necessary conditions in terms of C_0-semigroups for the existence and summability of a discrete α-resolvent family. It is notable that we can give an explicit form of such family in terms of functions of probability, like the Lévy process $f_{s,\alpha}(t)$ and the Poisson distribution $p_n(t) = \frac{t^n e^{-t}}{n!}$. Observe that they play the role of sampling of the given semigroup.

Theorem 3.5. *Let $0 < \alpha < 1$ and A be the generator of an exponentially stable C_0-semigroup $\{T(t)\}_{t\geq 0}$ defined on a Banach space X. Then A generates a discrete α-resolvent family $\{S_\alpha(n)\}_{n\in\mathbb{N}_0}$ defined by*

$$S_\alpha(n)x := \int_0^\infty \int_0^\infty e^{-t}\frac{t^n}{n!} f_{s,\alpha}(t)T(s)x\, ds\, dt, \quad n\in\mathbb{N}_0, \quad x\in X. \tag{3.1}$$

Moreover, $\{S_\alpha(n)\}_{n\in\mathbb{N}_0}$ is summable.

Proof. Let $M>0$ and $\omega>0$ be such that $\|T(t)\|\leq Me^{-\omega t}$. Define

$$T_\alpha(t)x := \int_0^\infty f_{s,\alpha}(t)T(s)x\, ds, \quad t>0, \quad x\in X.$$

First, we observe that $T_\alpha(t)$ is well defined. Indeed, note that by (2.1) and Fubini's Theorem we obtain

$$\int_0^\infty e^{-\omega s}f_{s,\alpha}(t)\, ds = \frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty}\int_0^\infty e^{-\omega s}e^{zt}e^{-sz^\alpha}\, ds\, dz = \frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} e^{zt}(\int_0^\infty e^{-(z^\alpha+\omega)s}\, ds)\, dz.$$

Hence, by Cauchy's formula and (2.2) we get

$$\int_0^\infty e^{-\omega s}f_{s,\alpha}(t)\, ds = \frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} e^{zt}\frac{1}{z^\alpha+\omega}\, dz = t^{\alpha-1}E_{\alpha,\alpha}(-\omega t^\alpha). \tag{3.2}$$

This gives

$$\|T_\alpha(t)\| \leq \int_0^\infty f_{s,\alpha}(t)\|T(s)\|\, ds \leq M\int_0^\infty e^{-\omega s}f_{s,\alpha}(t)\, ds = Mt^{\alpha-1}E_{\alpha,\alpha}(-\omega t^\alpha), \tag{3.3}$$

and hence, again by (2.2) we obtain

$$\int_0^\infty e^{-Re(\lambda)t}\|T_\alpha(t)\|\, dt \leq M\int_0^\infty e^{-Re(\lambda)t}t^{\alpha-1}E_{\alpha,\alpha}(-\omega t^\alpha)\, dt = \frac{M}{[Re(\lambda)]^\alpha+\omega},$$

for $Re(\lambda) > |w|^{1/\alpha}$. Consequently, $T_\alpha(t)$ is Laplace transformable and, using Fubini's theorem, we obtain

$$\hat{T}_\alpha(\lambda)x := \int_0^\infty e^{-\lambda t}T_\alpha(t)x\, dt = \int_0^\infty (\int_0^\infty e^{-\lambda t}f_{s,\alpha}(t)\, dt)T(s)x\, ds.$$

Therefore, by (i) in Proposition 2.7 and a well-known property on the Laplace transform of C_0-semigroups, we have

$$(\lambda^\alpha I - A)\hat{T}_\alpha(\lambda)x = (\lambda^\alpha I - A)\int_0^\infty e^{-s\lambda^\alpha}T(s)x\, ds = x, \quad x\in X,$$

and

$$\hat{T}_\alpha(\lambda)(\lambda^\alpha I - A)x = \int_0^\infty e^{-s\lambda^\alpha}T(s)(\lambda^\alpha I - A)x\, ds = x, \quad x\in D(A).$$

It shows that A commutes with $T_\alpha(t)$ on $D(A)$ and

$$\hat{T}_\alpha(\lambda)x = \frac{1}{\lambda^\alpha}x + A\frac{1}{\lambda^\alpha}\hat{T}_\alpha(\lambda)x, \quad x\in X.$$

By inversion of the Laplace transform, we obtain the identity

$$T_\alpha(t)x = g_\alpha(t)x + A\int_0^t g_\alpha(t-s)T_\alpha(s)x\, ds, \quad x\in X, \tag{3.4}$$

and since A is closed, we also get

$$T_\alpha(t)x = g_\alpha(t)x + \int_0^t g_\alpha(t-s)T_\alpha(s)Ax\,ds, \quad x \in D(A).$$

Note that the for $\omega > 0$ and $0 < \alpha < 1$ the function $t \to t^{\alpha-1}E_{\alpha,\alpha}(-\omega t^\alpha)$ is integrable on $\mathbb{R}_+$, see [43, Lemma 3.9]. Since $e^{-t}\frac{t^n}{n!} \leq 1$ for all $t \geq 0$ and $n \in \mathbb{N}$ it follows from (3.3) that $S_\alpha(n)$ defined by (3.1) exists for all $n \in \mathbb{N}$ and is summable. In fact,

(3.5)
$$\sum_{n=0}^{\infty}\|S_\alpha(n)\| \leq \sum_{n=0}^{\infty}\int_0^\infty e^{-t}\frac{t^n}{n!}\|T_\alpha(t)\|\,dt = \int_0^\infty \|T_\alpha(t)\|\,dt \leq M\int_0^\infty t^{\alpha-1}E_{\alpha,\alpha}(-\omega t^\alpha)\,dt < \infty.$$

Finally, we prove that $S_\alpha(n)$ satisfies (i) and (ii) in Definition 3.1. Indeed, (i) follows from the fact that A commutes with $T_\alpha(t)$ and A is closed. To prove (ii), we note that from (3.4) we obtain

$$\begin{aligned}
S_\alpha(n)x &= \int_0^\infty e^{-t}\frac{t^n}{n!}[g_\alpha(t)x + A\int_0^t g_\alpha(t-s)T_\alpha(s)x\,ds]\,dt \\
&= \int_0^\infty e^{-t}\frac{t^n}{n!}g_\alpha(t)x\,dt + A\int_0^\infty e^{-t}\frac{t^n}{n!}\int_0^t g_\alpha(t-s)T_\alpha(s)x\,ds\,dt \\
&= k^\alpha(n)x + A\int_0^\infty\int_s^\infty e^{-t}\frac{t^n}{n!}g_\alpha(t-s)T_\alpha(s)x\,dt\,ds \\
&= k^\alpha(n)x + A\int_0^\infty\Big(\int_0^\infty e^{-(\tau+s)}\frac{(\tau+s)^n}{n!}g_\alpha(\tau)\,d\tau\Big)T_\alpha(s)x\,ds \\
&= k^\alpha(n)x + A\int_0^\infty e^{-s}\Big(\int_0^\infty \frac{e^{-\tau}}{n!}\sum_{j=0}^n\binom{n}{j}\tau^{n-j}s^j g_\alpha(\tau)\,d\tau\Big)T_\alpha(s)\,ds \\
&= k^\alpha(n)x + A\int_0^\infty e^{-s}\sum_{j=0}^n\frac{n!}{j!(n-j)!}s^j\Big(\int_0^\infty\frac{e^{-\tau}}{n!}\tau^{n-j}g_\alpha(\tau)\,d\tau\Big)T_\alpha(s)x\,ds \\
&= k^\alpha(n)x + A\int_0^\infty e^{-s}\sum_{j=0}^n\frac{s^j}{j!}\Big(\int_0^\infty e^{-\tau}\frac{\tau^{n-j}}{(n-j)!}g_\alpha(\tau)\,d\tau\Big)T_\alpha(s)x\,ds \\
&= k^\alpha(n)x + A\int_0^\infty e^{-s}\sum_{j=0}^n\frac{s^j}{j!}k^\alpha(n-j)T_\alpha(s)x\,ds \\
&= k^\alpha(n)x + A\sum_{j=0}^n k^\alpha(n-j)\Big(\int_0^\infty\frac{s^j}{j!}e^{-s}T_\alpha(s)x\,ds\Big) \\
&= k^\alpha(n)x + A\sum_{j=0}^n k^\alpha(n-j)S_\alpha(n)x,
\end{aligned}$$

where we have made repeated use of Fubini's theorem and the formula $\int_0^\infty e^{-\beta t}t^{\alpha-1}\,dt = \frac{\Gamma(\alpha)}{\beta^\alpha}$ valid for $\alpha, \beta > 0$. See [25, Formula 3.381(4)]. It proves the claim and the theorem. □

Remark 3.6. *A careful reading of the proof of Lemma 3.9 in* [43] *combined with the estimate* (3.5) *shows that for* $0<\alpha<1$ *we have*

$$\|S_\alpha\|_1=\sum_{n=0}^{\infty}\|S_\alpha(n)\|\leq \frac{1}{\omega\alpha\pi}\Big(\frac{\pi}{2}-\arctan(\cot(\alpha\pi))\Big)=\frac{1}{\omega}.$$

Using the Gearhart-Prüss-Greiner theorem that characterizes the uniform exponential stability of semigroups in Hilbert spaces (see [19, Chapter V, Theorem 1.11]), we obtain the following remarkable corollary.

Corollary 3.7. *Let* $0<\alpha<1$ *and* A *be the generator of a* C_0*-semigroup on a Hilbert space* H *such that* $\{\mu\in\mathbb{C}: Re(\mu)>0\}\subset\rho(A)$ *and satisfy*

$$\sup_{Re(\mu)>0}\|(\mu-A)^{-1}\|<\infty.$$

Then A *generates a discrete* α*-resolvent family which is, in addition, summable.*

Remark 3.8. *The same conclusion of Corollary 3.7 can be obtained with different spectral conditions on the generator* A *by assuming more regularity on the semigroup. For example, suppose that* A *is the generator of an analytic* C_0*- semigroup such that*

$$\sup_{\lambda\in\sigma(A)} Re(\lambda)<0,$$

where $\sigma(A)$ *denotes the spectrum of* A*, then the same conclusion of Corollary 3.7 holds. Moreover, we have the advantage that we can consider Banach spaces instead of only Hilbert. See* [41, Theorem 4.3, p.118].

4. Almost automorphic solutions for linear fractional difference equations

Let A be a closed linear operator with domain $D(A)$ defined on X. In this section, we consider the non-homogeneous linear fractional difference equation given by

$$\Delta^\alpha u(n)=Au(n+1)+f(n),\qquad n\in\mathbb{Z}, \tag{4.1}$$

for $0<\alpha<1$.

Remember from the previous section that $\rho(n)=|n|^{\alpha-1}, n\in\mathbb{Z}$.

Definition 4.1. *A sequence* $u\in l^1_\rho(\mathbb{Z},X)$ *is called a strong solution for equation* (4.1) *if* $u(n)\in D(A)$ *for all* $n\in\mathbb{Z}$ *and* u *satisfies* (4.1).

The following theorem is one of the main results of this paper. Observe that $l^1(\mathbb{Z},X)\subset l^1_\rho(\mathbb{Z},X)$ for $0<\alpha<1$.

Theorem 4.2. *Let* $0<\alpha<1$ *and* A *be the generator of a summable discrete* α*-resolvent family* $\{S_\alpha(n)\}_{n\in\mathbb{N}_0}\subset\mathcal{B}(X)$*. If* $f\in l^1(\mathbb{Z},D(A))$*, then*

$$u(n):=\sum_{j=-\infty}^{n-1}S_\alpha(n-1-j)f(j),\qquad n\in\mathbb{Z},$$

is a strong solution of (4.1). *Moreover* $u\in l^1(\mathbb{Z},X)$.

Proof. Note that u is well defined and $u\in l^1(\mathbb{Z},X)$ with $\|u\|_{l^1}\leq\|S_\alpha\|_1\|f\|_{l^1}$. Next, let $n\in\mathbb{Z}$ be fixed and note that $S_\alpha(m)f(n-1-m)\in D(A)$ for all $m\in\mathbb{N}_0$ by (iii) of Theorem 3.2. Also $f(j)\in D(A)$ for all $j\in\mathbb{Z}$ by hypothesis. Then

$$AS_\alpha(n-1-j)f(j)=S_\alpha(n-1-j)Af(j).$$

Using that A is a closed operator, we get that $u(n) \in D(A)$ for all $n \in \mathbb{Z}$. To finish we prove that u satisfies (4.1):

$$\begin{aligned}
\Delta^\alpha u(n) &= \Delta\Delta^{-(1-\alpha)}u(n) = \Delta^{-(1-\alpha)}u(n+1) - \Delta^{-(1-\alpha)}u(n)\\
&= \sum_{j=-\infty}^{n+1} k^{1-\alpha}(n+1-j)\sum_{i=-\infty}^{j-1} S_\alpha(j-1-i)f(i)\\
&- \sum_{j=-\infty}^{n} k^{1-\alpha}(n-j)\sum_{i=-\infty}^{j-1} S_\alpha(j-1-i)f(i)\\
&= \sum_{j=-\infty}^{n+1} k^{1-\alpha}(n+1-j)\sum_{i=-\infty}^{j-1}\Big(k^\alpha(j-1-i)f(i) + \sum_{l=0}^{j-1-i} k^\alpha(j-1-i-l)AS_\alpha(l)f(i)\Big)\\
&- \sum_{j=-\infty}^{n} k^{1-\alpha}(n-j)\sum_{i=-\infty}^{j-1}\Big(k^\alpha(j-1-i)f(i) + \sum_{l=0}^{j-1-i} k^\alpha(j-1-i-l)AS_\alpha(l)f(i)\Big).
\end{aligned}$$

It is easy to see that we can apply Fubini's Theorem to all above summands, where we use that $f \in l^1(\mathbb{Z}, D(A))$ and $\{S_\alpha(n)\}_{n\in\mathbb{N}_0}$ is summable. Then, after changes of variable, we get that

$$\begin{aligned}
\Delta^\alpha u(n) &= \sum_{i=-\infty}^{n} f(i)\sum_{j=i+1}^{n+1} k^{1-\alpha}(n+1-j)k^\alpha(j-1-i)\\
&- \sum_{i=-\infty}^{n-1} f(i)\sum_{j=i+1}^{n} k^{1-\alpha}(n-j)k^\alpha(j-1-i)\\
&+ \sum_{j=-\infty}^{n+1} k^{1-\alpha}(n+1-j)\sum_{i=-\infty}^{j-1}\sum_{v=i+1}^{j} k^\alpha(j-v)AS_\alpha(v-1-i)f(i)\\
&- \sum_{j=-\infty}^{n} k^{1-\alpha}(n-j)\sum_{i=-\infty}^{j-1}\sum_{v=i+1}^{j} k^\alpha(j-v)AS_\alpha(v-1-i)f(i)\\
&= \sum_{i=-\infty}^{n} f(i) - \sum_{i=-\infty}^{n-1} f(i) + \sum_{j=-\infty}^{n+1} k^{1-\alpha}(n+1-j)\sum_{v=-\infty}^{j} k^\alpha(j-v)Au(v)\\
&- \sum_{j=-\infty}^{n} k^{1-\alpha}(n-j)\sum_{v=-\infty}^{j} k^\alpha(j-v)Au(v)\\
&= f(n) + \sum_{v=-\infty}^{n+1} Au(v) - \sum_{v=-\infty}^{n} Au(v) = f(n) + Au(n+1),
\end{aligned}$$

where we have use the semigroup property of the kernel k^α and that $k^1(n) = 1$ for all $n \in \mathbb{N}_0$. □

From Corollary 3.7 we immediately get the following result.

Corollary 4.3. *Let $0 < \alpha < 1$ and A be the generator of a C_0-semigroup on a Hilbert space H such that $\{\mu \in \mathbb{C} : Re(\mu) > 0\} \subset \rho(A)$ and satisfy*

$$\sup_{Re(\mu)>0} \|(\mu - A)^{-1}\| < \infty.$$

Then for all $f \in l^1(\mathbb{Z}, D(A))$ *the equation* (4.1) *admits a strong solution* $u \in l^1(\mathbb{Z}, H)$.

Since our objective is to study the solubility of (4.1) in the space of almost automorphic sequences, where the forcing term f is only bounded, we need to introduce the next definition. It is important to observe that due to Theorem 4.2 the following definition is consistent, in the sense that when the forcing term is sufficiently regular, then a mild solution is indeed a strong one.

Definition 4.4. *Let* $0 < \alpha < 1$, A *be the generator of a discrete* α*-resolvent family* $\{S_\alpha(n)\}_{n\in\mathbb{N}_0} \subset \mathcal{B}(X)$, *and* $f : \mathbb{Z} \to X$. *The sequence*

$$u(n) := \sum_{j=-\infty}^{n-1} S_\alpha(n-1-j)f(j), \qquad n \in Z.$$

is called a mild solution for equation (4.1) *if* $m \to S_\alpha(m)f(n-m)$ *is summable on* $\mathbb{N}_0$, *for each* $n \in \mathbb{Z}$.

The following result is now an easy consequence of the previous construction.

Theorem 4.5. *Let* $0 < \alpha < 1$ *and* A *be the generator of a summable discrete* α*-resolvent family* $\{S_\alpha(n)\}_{n\in\mathbb{N}_0} \subset \mathcal{B}(X)$. *If* $f \in AA_d(\mathbb{Z}, X)$, *then*

$$u(n) = \sum_{j=-\infty}^{n-1} S_\alpha(n-1-j)f(j), \qquad n \in \mathbb{Z},$$

is an almost automorphic mild solution of (4.1).

Proof. First note that u is well defined since

$$\|u(n)\| \le \|S_\alpha\|_1 \|f\|_\infty,$$

for all $n \in \mathbb{Z}$. We also have $u \in AA_d(\mathbb{Z}, X)$ by [4, Theorem 2.13] and [4, Remark 2.14]. Then u is an almost automorphic mild solution of (4.1). □

5. Almost automorphic solutions for nonlinear fractional difference equations

In this section we focus in the study of solutions for the nonlinear fractional difference equation

$$\Delta^\alpha u(n) = Au(n+1) + f(n, u(n)), \qquad n \in \mathbb{Z}, \tag{5.1}$$

for $0 < \alpha < 1$, where A is the generator of a discrete α-resolvent family $\{S_\alpha(n)\}_{n\in\mathbb{N}_0} \subset \mathcal{B}(X)$. We want to find almost automorphic solutions for (5.1) when $f \in AA_d(\mathbb{Z} \times X, X)$, that is, f is almost automorphic in the first discrete variable.

Definition 5.1. *Let* $0 < \alpha < 1$, A *be the generator of a discrete* α*-resolvent family* $\{S_\alpha(n)\}_{n\in\mathbb{N}_0} \subset \mathcal{B}(X)$, *and* $f : \mathbb{Z} \times X \to X$. *We say that a sequence* $u \in s(\mathbb{Z}, X)$ *is a mild solution of* (5.1) *if* $m \to S_\alpha(m)f(n-m)$ *is summable on* $\mathbb{N}_0$, *for each* $n \in \mathbb{Z}$ *and* u *satisfies*

$$u(n+1) = \sum_{j=-\infty}^{n} S_\alpha(n-j)f(j, u(j)), \qquad n \in \mathbb{Z}.$$

Our first result in this section provides a simple criteria for existence and uniqueness of almost automorphic mild solutions.

Theorem 5.2. *Let $0 < \alpha < 1$ and A be the generator of a summable discrete α-resolvent family $\{S_\alpha(n)\}_{n\in\mathbb{N}_0} \subset \mathcal{B}(X)$. If $f \in AA_d(\mathbb{Z}\times X, X)$ and it is globally Lipschitz in the following sense:*

$$\|f(n,x) - f(n,y)\| \leq L\|x-y\|, \quad \text{for all } n \in \mathbb{Z}, \text{ and } x, y \in X,$$

where $L < \frac{1}{\|S_\alpha\|_1}$, then (5.1) *admits a unique almost automorphic mild solution.*

Proof. We consider the operator $T : AA_d(\mathbb{Z}, X) \to AA_d(\mathbb{Z}, X)$ defined by

$$(Tu)(n) := \sum_{j=-\infty}^{n-1} S_\alpha(n-1-j)f(j,u(j)), \quad n \in \mathbb{Z}. \tag{5.2}$$

Note that T is well defined by [4, Theorem 2.10] and [4, Theorem 2.13]. In addition, for $u, v \in AA_d(\mathbb{Z}, X)$ and $n \in \mathbb{Z}$ the following inequality holds,

$$\begin{aligned}\|(Tu)(n) - (Tv)(n)\| &\leq \sum_{j=-\infty}^{n-1} \|S_\alpha(n-1-j)(f(j,u(j)) - f(j,v(j)))\| \\ &\leq L\sum_{j=-\infty}^{n-1} \|S_\alpha(n-1-j)\|\|u(j)-v(j)\| \\ &\leq L\|S_\alpha\|_1\|u-v\|_\infty.\end{aligned}$$

By hypothesis we conclude that T is a contraction, and using Banach fixed point theorem we get that there exists a unique almost automorphic mild solution of (5.1) . □

Using Remark 3.6 we have a precise estimate for $\|S_\alpha\|_1$ that can be used to prove the following corollary.

Corollary 5.3. *Let $0 < \alpha < 1$ and A be the generator of a C_0-semigroup $T(t)$ such that $\|T(t)\| \leq Me^{-\omega t}$, for some $M > 0$ and $\omega > 0$. If $f \in AA_d(\mathbb{Z}\times X, X)$ is globally Lipschitz with constant*

$$L < \frac{1}{\omega}$$

then (5.1) *admits a unique almost automorphic mild solution.*

Proof. It is clear using Theorem 3.5 and the proof of Theorem 5.2. □

We end this section with a modified hypothesis on the previous Lipschitz condition, assuming local instead of global. See Corollary 2.6.

Theorem 5.4. *Let $0 < \alpha < 1$ and A be the generator of a summable discrete α-resolvent family $\{S_\alpha(n)\}_{n\in\mathbb{N}_0} \subset \mathcal{B}(X)$. Let $f : \mathbb{Z}\times X \to X$ be a discrete almost automorphic function in $k \in \mathbb{Z}$ for each $x \in X$ that satisfies a local Lipschitz condition. If there exist $r_0 > 0$ such that*

$$\|S_\alpha\|_1\Big(L(r_0) + \frac{\sup_k \|f(k,0)\|}{r_0}\Big) < 1$$

then (5.1) *admits a unique almost automorphic mild solution u with $\|u\|_\infty := \sup_k \|u(k)\| \leq r_0$.*

Proof. First we consider the operator $T : AA_d(\mathbb{Z}, X) \to AA_d(\mathbb{Z}, X)$ given by (5.2), that is well defined by Corollary 2.6 and [4, Theorem 2.13].

Let $B_{r_0}(0) := \{u \in AA_d(\mathbb{Z}, X) : \|u\|_\infty < r_0\}$ be the ball of radius r_0 on $AA_d(\mathbb{Z}, X)$. We show that $T\big(B_{r_0}(0)\big) \subset B_{r_0}(0)$. Indeed, let u be in $B_{r_0}(0)$. Since f is locally Lipschitz, we get that

$$\|f(k, u(k))\| \leq \|f(k, u(k)) - f(k, 0)\| + \|f(k, 0)\| \leq L(r_0)\|u(k)\| + \|f(k, 0)\|, \quad k \in \mathbb{Z}.$$

Moreover, we have the following estimate

$$\begin{aligned}
\|T(u)(n)\| &\leq \sum_{j=-\infty}^{n-1} \|S_\alpha(n-1-j)\|\|f(j, u(j)) - f(j, 0)\| + \sum_{j=-\infty}^{n-1} \|S_\alpha(n-1-j)\|\|f(j, 0)\| \\
&\leq L(r_0) \sum_{j=-\infty}^{n-1} \|S_\alpha(n-1-j)\|\|u(j)\| + \|S_\alpha\|_1 \sup_k \|f(k, 0)\| \\
&\leq \|S_\alpha\|_1 \Big(L(r_0) + \frac{\sup_k \|f(k, 0)\|}{r_0}\Big) r_0 \leq r_0,
\end{aligned}$$

proving the claim. On the other hand, for $u, v \in B_{r_0}(0)$ we have that

$$\begin{aligned}
\|Tu(n) - Tv(n)\| &\leq \sum_{j=-\infty}^{n-1} \|S_\alpha(n-1-j)\|\|f(j, u(j)) - f(j, v(j))\| \\
&\leq L(r_0) \sum_{j=-\infty}^{n-1} \|S_\alpha(n-1-j)\|\|u(j) - v(j)\| \\
&\leq \|S_\alpha\|_1 L(r_0) \|u - v\|_\infty
\end{aligned}$$

Observing that $\|S_\alpha\|_1 L(r_0) < 1$, it follows that T is a contraction in $B_{r_0}(0)$. Then there is a unique $u \in B_{r_0}(0)$ such that $Tu = u$.

□

The following corollary is an immediate consequence of the previous results.

Corollary 5.5. *Let $0 < \alpha < 1$ and A be the generator of a C_0-semigroup $T(t)$ such that $\|T(t)\| \leq Me^{-\omega t}$, for some $M > 0$ and $\omega > 0$. If $f \in AA_d(\mathbb{Z} \times X, X)$ is locally Lipschitz and satisfy*

$$\frac{1}{\omega}\Big(L(r_0) + \frac{\sup_k \|f(k, 0)\|}{r_0}\Big) < 1,$$

for some $r_0 > 0$, then (5.1) admits a unique almost automorphic mild solution.

6. Examples and Applications

In this section we give some examples and applications to illustrate how our abstract results apply.

Example 6.1. *We consider a population of cells that are distinguished by their individual size. We can describe the population at discrete time $k \in \mathbb{N}$ by the number $n(k, s)$ of cells having size s by means of the following evolution equation (see* [19, p.349] *and references* *therein)*

$$
\begin{aligned}
n(k-1,s)-n(k,s) &= -\frac{\partial}{\partial s}n(k,s)-\mu(s)n(k,s)-b(s)n(k,s) \\
&+ \begin{cases} 4b(2s)n(k,2s) & \beta/2 \le s \le 1/2 \\ 0 & 1/2 < s \le 1 \end{cases}
\end{aligned} \tag{6.1}
$$

with boundary condition $n(k-1,\beta/2)=0$, $k\in\mathbb{N}$ and initial condition $n(0,s)=n_0(s)$ for $\beta/2\le s\le 1$. Here $\beta>0$ denotes the minimal cell size, μ is a positive continuous function on $[\beta/2,1]$ corresponding to the death rate, and b is the division, being a continuous function satisfying $b(s)>0$ for $s\in(\beta,1)$ and $b(s)=0$ otherwise.

We rewrite (6.1) *as an abstract difference equation. As a natural Banach space we choose $X=L^1(\beta/2,1)$ in which the norm $\|f\|$ of a positive function is the size of the total cell population represented by f.*

We define

$$A_0 f := -f' - (\mu+b)f, \quad D(A_0) := \{f\in W^{1,1}(\beta/2,1) \ : \ f(\beta/2)=0\},$$

and

$$Bf(s) := \begin{cases} 4b(2s)f(2s) & \beta/2\le s\le 1/2 \\ 0 & 1/2<s\le 1 \end{cases}$$

Set $A:=A_0+B$ with $D(A)=D(A_0)$. With this definition, equation (6.1) *becomes the abstract difference equation*

$$\Delta u(k) = Au(k+1), \quad k\in\mathbb{N}_0,$$

where $u:\mathbb{N}_0\to L^1(\beta/2,1)$ is defined by $u(k)(s):=n(k,s)$. By [19, Chapter VI, Proposition 1.3]*, the operator $(A,D(A))$ generates a strongly continuous semigroup $\{T(t)\}_{t\ge 0}$ on X. Moreover, by* [19, Chapter VI, Corollary 1.17] *the semigroup $\{T(t)\}_{t\ge0}$ is positive on the Banach lattice $X:=L^1(\beta/2,1)$ and uniformly exponentially stable if and only if*

$$\xi(0) := -1 + \int_{\beta/2}^{1/2} 4b(2s)e^{-\int_\sigma^{2\sigma}(\mu(\tau)+b(\tau))d\tau}d\sigma < 0$$

See [19, Chapter VI, Theorem 1.19]*. By Theorem 3.5, we conclude that for each $0<\alpha<1$ the operator A generates a summable α-resolvent family $S_\alpha(n)$.*

Example 6.2. *Next, we consider the fractional difference equation*

$$\Delta^\alpha u(k) = Au(k+1) + \frac{\nu g(k)u(k)}{1+\sup_k|u(k)|}, \quad k\in\mathbb{Z}, \tag{6.2}$$

where $0<\alpha<1$ act as a tuning parameter for the difference equation (6.2)*, the operator A is the generator of an exponentially stable C_0-semigroup on a Banach space X and $g(k)$ is an almost automorphic function that play the role of harmonic oscillations covered with big noise* [45]*. The parameter $\nu>0$ is a control of the size of the such oscillation.*

Note that

$$f(k,x) = \frac{\nu g(k)x}{1+\|x\|_\infty}, \quad k\in\mathbb{Z}, \quad x\in X.$$

Hence, we have the estimate

$$\begin{aligned}
\|f(k,x)-f(k,y)\|_\infty &\leq \nu|g(k)|\|\frac{x}{1+\|x\|_\infty}-\frac{y}{1+\|y\|_\infty}\|_\infty \\
&\leq \nu|g(k)|\Big\|(x-y)\frac{1}{1+\|x\|_\infty}+y(\frac{1}{1+\|x\|_\infty}-\frac{1}{1+\|y\|_\infty})\Big\|_\infty \\
&\leq \nu|g(k)|\Bigg[\|x-y\|_\infty+\|y\|_\infty\Big|\|y\|_\infty-\|x\|_\infty\Big|\Bigg] \\
&\leq \nu\|g\|_\infty(1+\|y\|)\|x-y\|_\infty.
\end{aligned}$$

Therefore, we can choose

$$L(r)=\nu\|g\|_\infty(1+r),\quad r>0,$$

to deduce that $f(k,x)$ is locally Lipschitz. Since $f(k,0)=0$, we obtain that for sufficiently small ν the condition

$$\|S_\alpha\|_1 L(r)<1$$

is satisfied. We conclude, by Theorem 5.4, that the fractional model (6.2) *admits a unique almost automorphic mild solution.*

Example 6.3. *Let $0<\alpha<1$. We consider the fractional difference scalar equation*

$$\Delta^\alpha u(n)=\lambda u(n+1)+f(n,u(n)),\quad n\in\mathbb{Z},$$

where λ is a complex number with $Re(\lambda)<0$ and $f:\mathbb{Z}\times\mathbb{C}\to\mathbb{C}$. It is clear that λ is the generator of the exponentially stable C_0-semigroup $T(t)=e^{\lambda t}$, for $t\geq 0$. Then we apply Theorem 3.5, and we assert that λ is the generator of a summable discrete α-resolvent family $\{S_\alpha(n)\}_{n\in\mathbb{N}_0}$ given by

$$\begin{aligned}
S_\alpha(n) &= \int_0^\infty\int_0^\infty e^{-t}\frac{t^n}{n!}f_{s,\alpha}(t)e^{\lambda s}\,ds\,dt=\int_0^\infty e^{-t}\frac{t^n}{n!}t^{\alpha-1}E_{\alpha,\alpha}(\lambda t^\alpha)\,dt \\
&= \frac{(-1)^n}{n!}\Big(\mathcal{L}(t^{\alpha-1}E_{\alpha,\alpha}(\lambda t^\alpha))\Big)^{(n)}(1)=\frac{(-1)^n}{n!}\Big((s^\alpha-\lambda)^{-1}\Big)^{(n)}_{|s=1},
\end{aligned}$$

where we have used (3.2) *and* (2.2) *and $\mathcal{L}$ denotes Laplace transform.*

On the other hand, by Theorem 3.2, the family $\{S_\alpha(n)\}_{n\in\mathbb{N}_0}$ has the representation given by $S_\alpha(0)=(1-\lambda)^{-1}$ and $S_\alpha(n)=\sum_{j=1}^{n}\beta_{\alpha,n}(j)(1-\lambda)^{-(j+1)}$, for $n\in\mathbb{N}_0$. Then we conclude that the coefficients $\beta_{\alpha,n}(j)$ can be obtained by means of $\Big((s^\alpha-\lambda)^{-1}\Big)^{(n)}_{|s=1}$.

Remark 6.4. *To finish, we mention that it is possible to apply the framework indicated in this paper to study more complexes classes of fractional difference equations defined on the scale $\mathbb{Z}$, taking advantage on the properties of the convolution form that we have presented in the definition of fractional difference considered in this paper and in* [35]. *One of the most interesting open problems to be done, relies on the extension of the ideas that we have indicated in this work to the case $1<\alpha\leq 2$. This is a not trivial task because it heavily depends on each difference equation to be studied. Also, it is worthwhile to comment that the theory presented is flexible to be adapted to the study of other qualitative behavior of solutions, like almost periodic sequences, asymptotically almost periodic sequences, and so on. We leave the study of such properties for future work.*

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